2011/35

Sufficient and necessary conditions for perpetual multi-assets exchange options

Joachim Gahungu and Yves Smeers

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

CORE DISCUSSION PAPER
2011/35

Sufficient and necessary conditions for perpetual multi-assets exchange options

Joachim GAHUNGU [1] and Yves SMEERS[2]

July 2011

Abstract

This paper considers the general problem of optimal timing of the exchange of the sum of *n* Ito-diffusions for the sum of *m* others (e.g., the optimal time to exchange a geometric Brownian motion for a geometric mean reverting process). We first contribute to the literature by providing analytical sufficient conditions and necessary conditions for optimal stopping (i.e. sub- and super- sets of the stopping region) for some sub-cases of the general problem. We then exhibit a connection between the problem of finding sufficient conditions for optimal stopping and linear programming. This connection provides a unified approach which does not only allow to recover previous analytically determinable subsets of the stopping region, but also allows to characterize (more complex) subsets of the stopping region that do not have an analytical expression. In the particular case where all assets are geometric Brownian motions, this connection gives us new insights. In particular, it simplifies the expression of the subset of the stopping region identified by Nishide and Rogers (2011). Our numerical examples finally confirms the good behavior of the candidate investment rule introduced by Gahungu and Smeers (2011) for this particular case, which seems to comfort a conjecture that their rule might be optimal.

Keywords: optimal stopping, stopping region, geometric Brownian motion, geometric mean reverting process, Schwartz process.

JEL Classification: D81, G11

---

[1] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium.
E-mail: Joachim.gahungu@ulg.ac.be.
[2] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium.
E-mail: yves.smeers@ulg.ac.be. This author is also member of ECORE, the association between CORE and ECARES.

The authors are thankful to Nicolas Gillis for his several remarks and suggestions that helped to improve this paper. Financial support from GDF-Suez is gratefully acknowledged.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

A basic generic problem encountered in quantitative finance is the determination of the right time to exchange uncertain assets. For example, this problem is encountered in pricing and optimal exercise of American securities. It is also encountered in the theory of investment under uncertainty. The related mathematical model is optimal stopping, and here we consider optimal stopping problems inspired by the following particular type of exchange: Suppose an investor willing to optimally exchange an asset having a reverting trend for an asset having an explosive trend. When is the right time to proceed to such exchange? Thus in the type of exchange we consider there are several assets and they have different continuous dynamical properties.

1.1 Mathematical setup

The mathematical framework is the following. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space and $X^x(t,\omega) : \Omega \times \Re_+ \to \Re^{n+m}$ be a $(n+m)$-dimensional Ito-diffusion starting at $x \in \Re^{n+m}$. We require that each component of X satisfies a one-dimensional stochastic differential equation (SDE) of the form

$$\begin{cases} \mathrm{d}X_i(t,\omega) = \mu_i(X_i)\mathrm{d}t + \sigma_i(X_i)\mathrm{d}\mathbb{B}_i(t,\omega) \\ X_i(0,\omega) = x_i \quad \text{a.s.} \quad \mathbb{P}^x \end{cases} \qquad i = 1, \dots, n+m, \tag{1}$$

where $\mathbb{B}_i : \Omega \times \Re_+ \to \Re$ is a one dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ for each $i = 1, \dots, n+m$. We note $\rho_{ij} = \mathrm{corr}(\mathbb{B}_i, \mathbb{B}_j)$ the correlation between two driving Brownian motions. We often use the following less explicit notation $X^x(t,\omega) = X(t,\omega) = X_t(\omega) = X_t$ when working on the entire vector X (and not on a specific component). If we work on a specific component, say X_j, of X we write $X_j^x(t,\omega) = X_j(t,\omega) = X_j(t) = X_j^t$. Throughout the paper, x, t and τ will always refer to initial position, time and random time respectively. Let $\mathbb{E}^x$ denotes the expectation w.r.t. the probability law $\mathbb{P}^x$ generated by the stochastic process $X^x(t,\omega)$ since its departure from x, and $\mathcal{S}$ the set of stopping times. This paper aims at characterizing the stopping region of the discounted optimal stopping problem:

Problem 1 (The (n,m) exchange)**.**

$$\tau^\star(x,\omega) = \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x\left[e^{-r\tau} g(X_\tau)\right], \tag{2}$$

$$g(x) \triangleq \sum_{i=1}^{n} x_i - \sum_{j=n+1}^{n+m} x_j. \tag{3}$$

In the terminology of optimal stopping problems, $g : \Re^{n+m} \to \Re$ is called the reward function. Given a stopping rule τ the value $\mathbb{E}^x[e^{-r\tau} g(X_\tau)]$ is called the performance of τ and the optimal performance $\mathbb{E}^x[e^{-r\tau^\star} g(X_{\tau^\star})]$ is called the value function. The stopping region of Problem 1 is noted $\mathbb{S}_{n,m}$.

The numbers n and m respectively indicate the numbers of income and cost streams. In finance, Problem 1 is referred to as the *optimal exercise of a perpetual $n+m$ basket option* or as the *optimal exchange of m assets for n others* (that we abbreviate by the generic term (n,m) *exchange*). We use the two terminologies depending on the circumstances. We define the sets $I \triangleq \{1, \dots, n\}$ (the indices of the income streams) and $J \triangleq \{n+1, \dots, n+m\}$ (the indices of the cost streams) in order to use the notational shortcut $g(x) \triangleq \sum_I x_i - \sum_J x_j$. Abusing notation, we write $X_I \triangleq (X_i, i \in I) = (X_1, \dots, X_n)$ and $X_J \triangleq (X_j, j \in J) = (X_{n+1}, \dots, X_{n+m})$.

Let us stress that $X_t(\omega) : \Omega \times \Re_+ \to \Re^{n+m}$ is a multi-dimensional diffusion thus a vector of possibly different one-dimensional diffusions. In practice, Problem 1 can serve as a mathematical model for an investor who exchanges assets having different continuous dynamical properties. An example is the $(1, 2)$ exchange where the price X_1 follows a geometric Brownian motion while the total cost is the sum of a standard Brownian motion with drift X_2 and a Schwartz process X_3. The fact that the considered problem has heterogeneous stochastic processes makes that $\mathbb{S}_{n,m}$ may have a very pathological shape. To the best of our knowledge, there is still no literature treating multi-asset optimal stopping problem with heterogeneous assets. We suspect this is due to the high intractability of multi-asset optimal stopping problems: even in the homogeneous case where all assets are geometric Brownian motions, the stopping region of the problem is not exactly identifiable, only approximations (subsets and supersets of the stopping regions) are available (see Section 2.1). In this paper, arguments developed for this particular case are adapted to more general exchanges, so as to determine similar approximations.

1.2 Structure of the paper

The paper is structured as follows. Section 2 summarizes what we know on optimal exchange of geometric Brownian motions and introduce the *reward decomposition technique* which is the main ingredient of several of our results. Section 3 gives the class of Ito diffusions for which our results are valid.

Sections 4 and 5 give analytical sufficient conditions for optimal stopping of Problem 1. Section 4 gives a general result which is a starting point, but we motivate that this sufficient condition may be too strong in exchange problems involving several geometric Brownian motions. In Section 5 we give finer results for two particular cases of this type.

In Section 6 we connect systematically the problem of determining a sufficient condition for optimal stopping (with respect to a certain reward decomposition) with the one of determining the emptiness of a polyhedron. In other words, we formulate sufficient conditions as $\mathcal{P}_x \neq \emptyset \Rightarrow x \in \mathbb{S}_{n,m}$ with $\mathcal{P}_x$ a polyhedron.

Section 7 gives necessary conditions for optimal stopping and a Table summarizing the results. Section 8 gives numerical examples and Section 9 concludes.

2 Background

2.1 A particular case: assets are geometric Brownian motions

The aforementioned particular instance of Problem 1 where the process X in (1) is a $n + m$ dimensional geometric Brownian motion

$$X_0 = x; \quad \mathrm{d}X_i(t, \omega) = \mu_i X_i(t)\mathrm{d}t + \sigma_i X_i(t)\mathrm{d}\mathbb{B}_i(t, \omega) \tag{4}$$

with $i = 1, \dots, n + m$ and μ_i, $\sigma_i \in \Re_+^{n+m}$ is interesting. It remains unsolved so far, but several authors provided partial results. Since one of these previous works will provide the main material to work on the general problem and that we will provide new insight on this particular case, we give a brief overview of properties regarding this homogeneous configuration.

I. The value function is linearly homogeneous and convex (see Olsen and Stensland, 1992). Consequently the stopping region $\mathbb{S}_{n,m}$ is a convex set.

II. The problem is solvable in the case $n = m = 1$ (see McDonald and Siegel, 1986): if X_i and X_j are two geometric Brownian motions the stopping region of

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x_i, x_j} \left[e^{-r\tau} \left(X_i^\tau - X_j^\tau \right) \right] \tag{5}$$

is given by

$$\mathbb{S}_{1,1} = \left\{ (x_i, x_j) \in \Re_+^2 : x_i \geq \gamma_{ij} x_j \right\}$$

where

$$\gamma_{ij} \triangleq \frac{\beta_{ij}^+}{\beta_{ij}^+ - 1} \tag{6}$$

$$\beta_{ij}^\pm \triangleq \left(\frac{1}{2} - \frac{\mu_{ij}}{\sigma_{ij}^2} \right) \pm \sqrt{ \left(\frac{1}{2} - \frac{\mu_{ij}}{\sigma_{ij}^2} \right)^2 + \frac{2(r - \mu_j)}{\sigma_{ij}^2} } \tag{7}$$

$$\sigma_{ij}^2 \triangleq \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2 \tag{8}$$

$$\mu_{ij} \triangleq \mu_i - \mu_j \tag{9}$$

(recall that $\rho_{ij} \triangleq \text{corr}(\mathbb{B}_i, \mathbb{B}_j) = \mathbb{E}\left(\text{d}\mathbb{B}_i \cdot \text{d}\mathbb{B}_j \right)/dt$).

III. For problems of higher dimensions, there is today no exact characterization of the stopping region: known results either identify (strict) sub- and super- sets of its optimal stopping region (see Olsen and Stensland, 1992, Hu and Øksendal, 1998 and Nishide and Rogers, 2011) or they provide a candidate stopping rule but cannot guarantee its optimality (see Gahungu and Smeers, 2011). These results are precisely the following.

   (a) Nishide and Rogers (2011) prove the sufficient condition for optimal stopping:

$$\mathbb{S}_{n,m} \supset \mathbb{S}_{n,m}^- \tag{10}$$

   with

$$\mathbb{S}_{n,m}^- \triangleq \text{conv} \left(\bigcup_{\substack{i \in I \\ j \in J}} A_{ij} \right) \tag{11}$$

   where

$$A_{ij} \triangleq \left\{ x \in \Re_+^{n+m} |\, x_i \geq \gamma_{ij} x_j,\, x_k = 0 \quad \text{for} \quad k \neq i, j \right\}, \tag{12}$$

   conv(A) stands for *the convex hull*[1] of the set A and γ_{ij} is given by (6). This result extends sufficient conditions derived for $(1, m)$ and $(n, 1)$ exchanges by

[1] The convex hull conv(S) of a set S is

$$\text{conv}(S) \triangleq \left\{ \sum_{i=1}^n c_i x_i \;\middle|\; \sum_{i=1}^n c_i = 1 \quad ; \quad x_i \in S \quad \text{and} \quad c_i \geq 0 \quad \forall i = 1, \ldots, n, \quad \forall n \in \mathbb{N} \right\}.$$

Following this definition, (11) is equivalent to

$$\mathbb{S}_{n,m}^- = \left\{ \sum_{\substack{i \in I \\ j \in J}} c_{ij} x_{ij} \;\middle|\; \sum_{\substack{i \in I \\ j \in J}} c_{ij} = 1 \quad ; \quad x_{ij} \in A_{ij} \quad \text{and} \quad c_{ij} \geq 0 \quad \forall i, j \in I, J \right\}.$$

Olsen and Stensland (1992) to more general (n, m) exchanges. However the mechanics used to prove the general result or the two particular cases are not alike.

To obtain sufficient conditions for optimal stopping of the $(1, m)$ and $(n, 1)$ exchanges, Olsen and Stensland (1992) rely on a reward decomposition argument that will be exposed in the next section. In contrast, the proof of (10) relies on the following simple argument: it is clear that $\mathbb{S}_{n,m}$ contains all the A_{ij}'s for $i \in I$ and $j \in J$, thus it contains $\bigcup_{i\in I, j\in J} A_{ij}$. Since $\mathbb{S}_{n,m}$ is convex (see I.) and $\mathrm{conv}(\bigcup_{i\in I, j\in J} A_{ij})$ is the smallest convex set containing $\bigcup_{i\in I, j\in J} A_{ij}$, then (10) holds.

Note finally that it is generally not trivial to characterize convex hulls. There exist efficient algorithms to find the vertices of the convex hull of a finite set of points, but the problem here is to characterize the convex hull of a finite union of polyhedra. The simple use of the definition of convex hull[1] does not allow one to merely determine whether a given $x \in \Re_+^{n+m}$ belongs to $\mathbb{S}_{n,m}^-$, except in $(1, m)$ and $(n, 1)$ exchanges for which $\mathbb{S}_{n,m}$ has an easily computable analytic expression. Nishide and Rogers (2011) do not elaborate on that caveat. In this paper we will prove (10) by a reward decomposition argument, thereby providing a tractable method to verify whether a given point $x \in \Re_+^{n+m}$ belongs to $\mathbb{S}_{n,m}^-$ or not.

(b) Nishide and Rogers (2011) identify the necessary condition for optimal stopping

$$\mathbb{S}_{n,m} \subseteq \mathbb{S}_{n,m}^+(X_u, X_v) \tag{13}$$

with

$$\mathbb{S}_{n,m}^+(X_u, X_v) \triangleq \Bigg\{ x \in \Re_+^{n+m} : \\ \gamma_{v1}x_1 + \ldots + \gamma_{vn}x_n \geq \gamma_{vu}\left(\frac{x_{n+1}}{\gamma_{n+1,u}} + \ldots + \frac{x_{n+m}}{\gamma_{n+m,u}}\right) \Bigg\} \tag{14}$$

for any geometric Brownian motion X_u and X_v. This result extends necessary conditions for optimal stopping derived for $(1, m)$ and $(n, 1)$ exchanges by Hu and Øksendal (1998) to more general (n, m) exchanges.

(c) Gahungu and Smeers (2011) propose a candidate stopping region

$$\mathbb{S}^\diamond = \left\{ x \in \Re_+^{n+m} : x_1 \geq x_1^\diamond(x_2, \ldots, x_{n+m}) \right\} \tag{15}$$

where $x_1^\diamond(\cdot) : \Re_+^{n+m-1} \to \Re_+$ is a closed-form determinable function (we omit to give its expression in the interest of space; see Gahungu and Smeers, 2011, Proposition 3). They prove that $x_1^\diamond(\cdot)$ is linearly homogeneous. Unlike (11) and (14), $x_1^\diamond(x_{-1})$ depends on inter-price and inter-cost correlations (ρ_{ij} for $i, j \leq n$ and ρ_{ij} for $i, j > n$, respectively). Needless to say the optimal investment rule should involve these correlations. On the numerical examples they study, they always find

$$\mathbb{S}_{n,m}^- \subset \mathbb{S}_{n,m}^\diamond \subset \mathbb{S}_{n,m}^+(X_u, X_v). \tag{16}$$

Moreover, the performance associated to $\mathbb{S}_{n,m}^\diamond$ is never lower than the performance associated to $\mathbb{S}_{n,m}^-$ and $\mathbb{S}_{n,m}^+(X_u, X_v)$. It provides a higher performance in exchange problems with strong inter-price or inter-cost correlations. Despite these encouraging findings, there is no proof of the optimality of $x_1^\diamond(\cdot)$ which is left as an open research issue.

2.2 The reward decomposition technique

In this paper, we mainly exploit an idea used by Olsen and Stensland (1992) to obtain sufficient conditions for optimal stopping for $(1, m)$ and $(n, 1)$ exchanges of geometric Brownian motions. This argument is referred to as the reward decomposition technique throughout the paper. It will be used with a view to determining sufficient conditions and necessary conditions for optimal stopping of more general problems.

The substance of the reward decomposition technique is as follows. Decompose additively the reward $g(x)$, i.e. write $g(x) = \sum_{i=1}^{d} g_i(x)$ for some chosen family of functions g_i, $i = 1, \ldots, d$. Call $\mathbb{S}_i$ the stopping region of $\sup_{\tau\in\mathcal{S}} \mathbb{E}^x[g_i(X_\tau)]$ for $i = 1, \ldots, d$. It follows from the inequality

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x[g(X_\tau)] \leq \sum_{i=1}^{d} \sup_{\tau\in\mathcal{S}} \mathbb{E}^x[g_i(X_\tau)] \tag{17}$$

that

$$x \in \left(\bigcap_{i=1,\ldots,d} \mathbb{S}_i \right) \Rightarrow x \in \mathbb{S}_{n,m}.$$

Of course, the implementation of this mechanism to effectively work out sufficient conditions for optimal stopping is a little more complex and probably easier to expose through an example. Let us consider the problem originally treated by Olsen and Stensland (1992), i.e. the $(1, m)$ exchange of geometric Brownian motions.

Example 1 (The $(1, m)$ exchange of GBMs)**.** *Consider the $(1, m)$ exchange of geometric Brownian motions. We have in mind the reward decomposition*

$$g(x) = x_1 - x_2 - \ldots - x_{m+1} = \sum_{i=2}^{m+1} (c_i x_1 - x_i) \triangleq \sum_{i=2}^{m+1} g_i(x) \tag{18}$$

for any c such that $\sum_{i=2}^{m+1} c_i = 1$.

i) Let $x \in \Re_+^{1+m}$. Suppose there exists $(c_2, \ldots, c_{m+1}) \in \Re_+^m$ such that

$$\sum_{i=2}^{m+1} c_i = 1, \tag{19}$$

$$c_i x_1 \geq \gamma_{1j} x_j \qquad j = 2, \cdots, m+1. \tag{20}$$

Then using the decomposition (18), (17) yields

$$\sup_{\tau} \mathbb{E}^x[g(X_\tau)] \leq \sum_{i=1}^{d} \sup_{\tau\in\mathcal{S}} \mathbb{E}^x[g_i(X_\tau)] = g(x) \tag{21}$$

thus it is optimal to stop at x.

ii) We now try to construct a condition on x such that there exists a $(c_2, \ldots, c_{m+1}) \in \Re_+^m$ satisfying (19) and (20). Define the candidate

$$\tilde{c}_j = \gamma_{1j} x_j / x_1, \qquad j = 2, \ldots, m$$

$$\tilde{c}_{m+1} = 1 - \sum_{j=2}^{m} \tilde{c}_j.$$

Then if $\tilde{c}_{m+1}x_1 \geq \gamma_{1,m+1}x_{m+1}$—which turns out to be equivalent to the condition

$$x_1 \geq \gamma_{12}x_2 + \ldots + \gamma_{1,m+1}x_{m+1} \tag{22}$$

derived by Olsen and Stensland—the candidate $\tilde{c}$ is such a c and it is optimal to stop.

We see in Example 1 that the obtained sufficient condition corresponds to the chose decomposition (i.e. to the decomposition of g in the g_i's, $i = 1, \ldots, d$). Moreover the decomposition has to be done "intelligently"; in a fashion allowing one to compute $\mathbb{S}_i$ for all $i = 1, \ldots, d$. In general, the decomposition implies a dimension reduction (in Example 1, $g_i(x) = g_i(x_1, x_i) = c_i x_1 - x_i$ only depends on two components of the basket) so that we loose some interaction information when using (17) (in Example 1, Eq. (21) we have $\mathbb{E}^x[g_i(X_\tau)] = \mathbb{E}^{(x_1,x_i)}[g_i(X_\tau)]$). Thus, in general, sufficient conditions derived by a decomposition argument do not depend on the entire correlation matrix of the problem. Therefore they are typically too strong (not necessary).

A critical part of Example 1 is the step ii), i.e. the determination of the analytic condition on x ensuring that there exists a function $g_i(x)$—in the considered class of decomposition function given by (18)—such that $x \in (\bigcap_{i=1,\ldots,d} \mathbb{S}_i)$. We see that for certain problem configurations and reward decompositions—e.g. the $(1, m)$ and $(n, 1)$ exchange of geometric Brownian motions—one is able to analytically characterize subsets of the stopping region. A first part of this paper will provide similar analytic sufficient conditions for optimal stopping for heterogeneous baskets.

In practice however, analytic characterization of the stopping region is not necessary. Traders observe in real time the current value x of assets of their portfolios, thus an algorithmic method determining whether immediate stopping (at x) is optimal might be sufficient. In Example 1, one may not have (22) but simply check if, given a point $x \in \Re_+^{1+m}$, there exists a $c \in \Re_+^m$ such that (19) and (20) hold. In other words, if the polyhedron

$$\mathcal{P}_x = \left\{ c \in \Re_+^m \;\middle|\; \begin{array}{ll} \sum_{j=2}^{m+1} c_j = 1, & \\ c_j x_1 \geq \gamma_{1j} x_j & j = 2, \ldots, 1+m \end{array} \right\}, \tag{23}$$

is not empty then it is optimal to stop at x. This expression of the sufficient condition is referred to as the *LP formulation* of condition (22). Needless to say the problem of determining whether a polyhedron is empty or not is easily handled by linear optimization solvers. Furthermore, note that this algorithmic approach remains applicable for more complex exchange configurations or more elaborated reward decomposition, cases for which one cannot hope finding analytical sufficient conditions.

For instance, the (n, m) exchange of geometric Brownian motion leads to the sufficient condition (10) which requires to characterize a convex hull which has no analytic expression. One particular contribution of this paper is to show that an alternative proof of (10) can be obtained via the reward decomposition technique. In other words there exists an LP formulation of the convex hull $\mathbb{S}_{n,m}^-$ (see Proposition 6) which turns out to be the most useful characterization of $\mathbb{S}_{n,m}^-$ in practice. Finally, regarding exercise of heterogeneous baskets, this algorithmic approach provides sufficient conditions for optimal stopping which are less strong than their analytic analogs (see e.g. Section 8.4).

Taking stock of this characterization of the reward decomposition technique, the structure of the paper can be restated more clearly as follows. We thus recall it here. Section 3 gives the class of Ito diffusions for which our results hold. In Section 4 we give a general analytic sufficient condition for optimal stopping. This condition can be refined for some cases where the basket contains several geometric Brownian motions; these refined sufficient conditions are given in Section 5. In Section 6 we systematically formulate sufficient conditions in the form $\mathcal{P}_x \neq \emptyset \Rightarrow x \in \mathbb{S}_{n,m}$ for polyhedras $\mathcal{P}_x$ more general than (23). We also come back to optimal exercise of baskets of geometric Brownian motions and find that the sufficient condition (10) derived by Nishide and Rogers (2011) can also be obtained via the reward decomposition technique; the latter being easier to use in practice. Section 7 gives a necessary condition for optimal stopping and a table summarizing the results, Section 8 gives numerical examples and Section 9 concludes.

3 Call and put class of diffusions

This section gives the class of Ito diffusions for which our results are valid. To each $X_i, i \in I$ (resp. $X_j, j \in J$) is associated the Problem 2 (resp. Problem 3) of optimal exercise of a perpetual call (resp. put) of exercise price $k \in \Re_+$.

Problem 2 (Perpetual American call option on X, of strike k)**.**

$$\tau^\star(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(X_\tau - k\right)\right] \tag{24}$$

Problem 3 (Perpetual American put option on X, of strike k)**.**

$$\tau^\star(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(k - X_\tau\right)\right] \tag{25}$$

In the following, Problem 2 (resp. Problem 3) is simply noted $[X-k]$ (resp. $[k-X]$) and its stopping region $\mathbb{S}[X-k]$ (resp. $\mathbb{S}[k-X]$). The results on Problem 1 presented in Sections 4 and 7 do not rely on analytic arguments (on path properties and stochastic differential equations of processes X_I and X_J) but on geometric comparisons between $\mathbb{S}_{n,m}$, $\mathbb{S}[X_i-k], i \in I$ and $\mathbb{S}[k-X_j], j \in J$. The only sufficient condition for these results to hold is that for all $i \in I$ and for all $j \in J$, Problem 2 on X_i and Problem 3 on X_j are "well defined" in the following sense.

Definition 1 ($\mathcal{C}$ class for Ito diffusions)**.** *X belongs to the* Call class*—and we note $X \in \mathcal{C}$—if and only if there exists a unique invertible function $C_X(\cdot) : \Re_+ \to \Re$ such that, $\forall k \in \Re^+$, $\mathbb{S}[X-k] = \{x \geq C_X(k)\}$.*

Definition 2 ($\mathcal{P}$ class for Ito diffusions)**.** *X belongs to the* Put class*—and we note $X \in \mathcal{P}$—if and only if there exists a unique invertible function $P_X(\cdot) : \Re_+ \to \Re$ such that, $\forall k \in \Re^+$, $\mathbb{S}[k-X] = \{x \leq P_X(k)\}$.*

The key notion in the definition of the $\mathcal{C}$ and $\mathcal{P}$ classes is the invertibility of the trigger function. If $X \in \mathcal{C}$, then $C_X : \Re_+ \to \Re$ is such that $\forall x \in \mathrm{Im}(C_X)$, there exists a unique $k \in \Re_+$ such that $C_X(k) = x$. Thus there exists $C_X^{-1} : \mathrm{Im}(C_X) \to \Re_+$ s.t. $k = C_X^{-1}(x)$. In other words, for any x belonging to $\mathrm{Im}(C_X)$, one can find a strike k such that x belongs to the frontier of the stopping region.

The following examples show that the classes $\mathcal{C}$ and $\mathcal{P}$ are not too restrictive. They both contain at least the following four processes: the standard Brownian motion, the geometric Brownian motion, the geometric mean reverting process and the Schwartz processes.

Example 2 (X is a Brownian motion with drift). *Suppose that*

$$\mathrm{d}X_t^x(\omega) = \mu \mathrm{d}t + \sigma \mathrm{d}\mathbb{B}_t(\omega), \qquad t \geq 0 \tag{26}$$

whose solution is trivially $X_t = x + \mu t + \sigma \mathbb{B}_t(\omega)$. *It is easy to show that* $\mathbb{S}[X-k] = \{x \geq k + 1/a^+\}$ *with* $a^+ > 0$ *(resp.* $\mathbb{S}[k-X] = \{x \leq k + 1/a^-\}$ *with* $a^- < 0$*) where* $a^{\pm} = -\mu \pm \sqrt{\mu^2 + 2\sigma^2 r}$. *Thus,* $X \in \mathcal{C}$ *and* $X \in \mathcal{P}$.

Example 3 (X is a geometric Brownian motion.). *If* $X_t(\omega)$ *is the* (μ, σ) *geometric Brownian motion*

$$\mathrm{d}X_t^x(\omega) = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}\mathbb{B}_t(\omega), \qquad t, x \geq 0 \tag{27}$$

and $0.5\sigma^2 < \mu < r$, *the solution of Problem 2 (resp. Problem 3) is* $\mathbb{S}[X-k] = \{x \geq \gamma^+ k\}$ *for* $\gamma^+ > 1$ *(resp.* $\mathbb{S}[k-X] = \{x \leq \gamma^- k\}$ *for* $\gamma^- < 1$*) with*

$$\gamma^{\pm} = \frac{\beta^{\pm}}{\beta^{\pm} - 1} \tag{28}$$

where

$$\beta^{\pm} = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

(see McDonald and Siegel, 1986). Thus, $X \in \mathcal{C}$ *and* $X \in \mathcal{P}$.

For these two examples, the trigger function is linear in k. This is not the case in general as illustrated in the two following examples.

Example 4 (X is a Schwartz process). *If* $X_t(\omega)$ *is the Schwartz process*

$$\mathrm{d}X_t^x(\omega) = \eta \left(\bar{X} - \log X_t\right) X_t \mathrm{d}t + \sigma X_t \mathrm{d}\mathbb{B}_t(\omega), \qquad t, x \geq 0 \tag{29}$$

then $\mathbb{S}[X-k] = \{x \geq t^+(k)\}$ *(resp.* $\mathbb{S}[k-X] = \{x \leq t^-(k)\}$*) for a certain* $t^+(k)$ *(resp.* $t^-(k)$*) which is not analytically computable. However, it is not hard to show that* $t^+(k)$ *(resp.* $t^-(k)$*) is unique and continuous and increasing in* k *(see Boyarchenko and Levendorskii, 2007, Chapter 14). Thus,* $X \in \mathcal{C}$ *and* $X \in \mathcal{P}$.

Example 5 (X is a geometric mean reverting process). *If* $X_t(\omega)$ *is the geometric mean reverting process*

$$\mathrm{d}X_t^x(\omega) = \eta \left(\bar{X} - X_t\right) X_t \mathrm{d}t + \sigma X_t \mathrm{d}\mathbb{B}_t(\omega), \qquad t, x \geq 0 \tag{30}$$

then $\mathbb{S}[X-k] = \{x \geq t^+(k)\}$ *(resp.* $\mathbb{S}[k-X] = \{x \leq t^-(k)\}$*) for a certain* $t^+(k)$ *(resp.* $t^-(k)$*) which is not analytically computable. However, there exist values of the parameters* η, $\bar{X}$ *and* σ *for which* $t^+(k)$ *(resp.* $t^-(k)$*) is unique, continuous and increasing in* k *(see Metcalf and Hassett, 1995 and Dixit and Pindyck, 1994). Thus,* $X \in \mathcal{C}$ *and* $X \in \mathcal{P}$.

4 Sufficient conditions for optimal stopping: a general result

4.1 The general result

The first main result of this paper is a sufficient condition for optimal stopping of Problem 1.

Proposition 1 (Sufficient condition for optimal stopping - general case)**.** *Assume*

1. $X_i \in \mathcal{C}, \forall i \in I$
2. $X_j \in \mathcal{P}, \forall j \in J$.

Then

$$\mathbb{S}_{n,m} \supseteq \mathbb{S}_{n,m}^{--} \triangleq \left\{ x \in \Re^{n+m} : \sum_I C_i^{-1}(x_i) \geq \sum_J P_j^{-1}(x_j) \right\}. \tag{31}$$

Proof. The proof follows from the equality

$$\sum_I X_i - \sum_J X_j = \sum_I (X_i - c_i) + \sum_J (c_j - X_j)$$

for any $c \in \Re_+^{n+m}$ s.t.

$$\sum_I c_i - \sum_J c_j = 0. \tag{32}$$

Using this decomposition, the proof follows the reward decomposition technique.

i) Let $x \in \Re^{n+m}$. Suppose there exists $c \in \Re_+^{n+m}$ such that (32) holds and

$$x_i \geq C_i(c_i) \quad \forall i \in I, \tag{33}$$

$$x_j \leq P_j(c_j) \quad \forall j \in J. \tag{34}$$

Then $x \in \mathbb{S}_{n,m}$ since (using the decomposition)

$$\begin{aligned}
&\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(\sum_I X_i^\tau - \sum_J X_j^\tau \right) \right] \\
&= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(\sum_I (X_i^\tau - c_i) + \sum_J (c_j - X_j^\tau) \right) \right] \\
&\leq \sum_I \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} (X_i^\tau - c_i) \right] + \sum_J \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} (c_j - X_j^\tau) \right] \\
&= \sum_I (x_i - c_i) + \sum_J (c_j - x_j) = g(x).
\end{aligned}$$

ii) It remains to find a condition on x such that one can find a c such that (32), (33) and (34) hold. Here we need the invertibility of $C_i, i \in I$ and $P_j, j \in J$. Define a candidate $\tilde{c}$ for c by

$$\begin{aligned}
\tilde{c}_i &\triangleq C_i^{-1}(x_i) \quad i \in I \setminus \{1\} \\
\tilde{c}_j &\triangleq P_j^{-1}(x_j) \quad j \in J \\
\tilde{c}_1 &\triangleq \sum_J \tilde{c}_j - \sum_{I \setminus \{1\}} \tilde{c}_i.
\end{aligned}$$

If $x_1 \geq C_1(\tilde{c}_1)$ i.e. if $x_1 \geq C_1 \Big(\sum_J P_j^{-1}(x_j) - \sum_{I \setminus \{1\}} C_i^{-1}(x_i) \Big)$ which is in fact (31), then the candidate $\tilde{c}$ is such a c. Consequently, $x \in \mathbb{S}_{n,m}$, which completes the proof.

□

The equation (31) is a sufficient condition for optimal stopping for Problem 1: for any $x \in \Re^{n+m}$ such that (31) holds, it is optimal to stop immediately and collect the reward $g(x) = \sum_I x_i - \sum_J x_j$. This condition is however not necessary i.e. it could have been optimal to stop sooner.

The next result explains how the subset $\mathbb{S}_{n,m}^{--}$ of the stopping region $\mathbb{S}_{n,m}$ shifts under the transformation $g(x) = \sum_I x_i - \sum_J x_j \to \tilde{g}(x) = \sum_I u_i x_i - \sum_J u_j x_j$, for u_i, $u_j > 0$. This result will turn out to be useful later on.

Corrolary 1. *Under the assumptions 1. and 2. of Proposition 1, the stopping region $\mathbb{S}_{n,m}^u$ of the problem*

$$\tau^\star(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(\sum_I u_i x_i - \sum_J u_j x_j \right) \right]$$

with $u_l > 0$ for $l \in I \cup J$ satisfies

$$\mathbb{S}_{n,m}^u \supseteq \mathbb{S}_{n,m}^{u--} \triangleq \left\{ x \in \Re^{n+m} : \sum_I u_i C_i^{-1}(x_i) \geq \sum_J u_j P_j^{-1}(x_j) \right\}. \tag{35}$$

Proof. The proof follows easily from the linear homogeneity of the reward function. For all $i \in I$, if $u_i \neq 0$, $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[(u_i X_i^\tau - k)e^{-r\tau}] = u_i \sup_{\tau \in \mathcal{S}} \mathbb{E}^x[(X_i^\tau - k/u_i)e^{-r\tau}]$. Thus, $\mathbb{S}[u_i X_i - k] = \mathbb{S}[X_i - k/u_i] = \{x_i : x_i \geq C_i(k/u_i)\}$. Define now $\tilde{C}_i(\cdot) \triangleq C_i(\cdot/u_i)$ whose inverse given by $\tilde{C}_i^{-1}(\cdot) = u_i C_i^{-1}(\cdot)$ is clearly well defined. Use the same arguments on $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[(k - u_j X_j^\tau)e^{-r\tau}]$ to define $\tilde{P}_j(\cdot) \triangleq P_j(\cdot/u_j)$ with $\tilde{P}_j^{-1} = u_j P_j^{-1}(\cdot)$, $\forall j \in J$. Following the proof of Proposition 1 having replaced C_i by $\tilde{C}_i$ and P_j by $\tilde{P}_j$ leads to (35). □

Example 6. *For the two assets case*

$$\tau^\star(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(u_1 X_1^\tau - u_2 X_2^\tau \right) \right],$$

Corollary 1 gives the sufficient condition

$$S_{1,1}^{u--} = \left\{ x \in \Re^2 : u_1 C_1^{-1}(x_1) \geq u_2 P_2^{-1}(x_2) \right\}. \tag{36}$$

4.2 The basket only contains GBMs - Part 1

Consider baskets exclusively composed of geometric Brownian motions. In this particular case, Proposition 1 takes the following form.

Corrolary 2 (Sufficient condition for optimal stopping - Basket of GBMs)**.** *Assume that the basket exclusively contains geometric Brownian motions i.e. that X_l satisfies (27) for all $l \in I \cup J$. Note γ_l the γ defined by* (28) *associated to the GBM X_l. Then $\forall i \in I$, $C_i(k) = \gamma_i^+ k$, $\forall j \in J$, $P_j(k) = \gamma_j^- k$ and*

$$\mathbb{S}_{n,m} \supseteq \mathbb{S}_{n,m}^{--} \triangleq \left\{ x \in \Re_+^{n+m} : \sum_I \frac{x_i}{\gamma_i^+} \geq \sum_J \frac{x_j}{\gamma_j^-} \right\}. \tag{37}$$

Recall that we know from Nishide and Rogers (2011) than in this particular case the optimal stopping region $\mathbb{S}_{n,m}$ also satisfies

$$\mathbb{S}_{n,m}^- \subset \mathbb{S}_{n,m} \subseteq \mathbb{S}_{n,m}^+(X_u, X_v) \tag{38}$$

where $\mathbb{S}^-_{n,m}$ and $\mathbb{S}^+_{n,m}(X_u, X_v)$ are respectively given by (10) and (14), with X_u and X_v two arbitrary geometric Brownian motion processes. The following result relates $\mathbb{S}^{--}_{n,m}$ to $\mathbb{S}^-_{n,m}$ for particular values of n and m.

Proposition 2. *$\mathbb{S}^{--}_{1,m} \subseteq \mathbb{S}^-_{1,m}$ and $\mathbb{S}^{--}_{n,1} \subseteq \mathbb{S}^-_{n,1}$.*

Proof. Since $\mathbb{S}_{1,1} = \{x \in \Re^2_+ : x_1 \geq \gamma_{12} x_2\}$ and $\mathbb{S}^{--}_{1,1} \subseteq \mathbb{S}_{1,1}$ one has $\gamma_{ij} \leq \gamma_i^+/\gamma_i^-$. Defining

$$B_{ij} \triangleq \left\{ x \in \Re^{n+m}_+ : \quad x_i \geq \frac{\gamma_i^+}{\gamma_j^-} x_j \quad , \quad x_k = 0 \quad \text{for} \quad k \neq i, j \right\} \tag{39}$$

one has $B_{ij} \subseteq A_{ij}$, $i \in I$, $j \in J$. But in the two particular cases where $n = 1$ or $m = 1$ one has $\mathbb{S}^{--}_{n,m} = \operatorname{conv}(\bigcup_{i \in I, j \in J} B_{ij})$ which leads to

$$\mathbb{S}^{--}_{n,m} = \operatorname{conv}\left(\bigcup_{\substack{i \in I \\ j \in J}} B_{ij} \right) \subseteq \operatorname{conv}\left(\bigcup_{\substack{i \in I \\ j \in J}} A_{ij} \right) = \mathbb{S}^-_{n,m}.$$

□

It is difficult to work out such result for general values of n and m because the convex hull of $\cup_{I,J} A_{ij}$ is not easy to manipulate algebraically. But we strongly suspect that $\mathbb{S}^{--}_{n,m} \subseteq \mathbb{S}^-_{n,m}$ because a reward decomposition by a sum of perpetual call and put options excessively relaxes the problem when the basket only contains GBMs: it overlooks the entire correlation matrix of the problem. In contrast, the decomposition of the reward by a sum of two assets exchange options incorporates the correlations between the prices and the costs (but excludes inter-price or inter-cost correlations). The next section precisely presents two particular configurations of the general (heterogeneous) problem for which one can find an analytic sufficient condition for optimal stopping which is better than $\mathbb{S}^{--}_{n,m}$, i.e. for which one can improve upon Proposition 1.

5 Sufficient conditions when the basket contains GBMs - Two particular cases

Proposition 1 is based on a reward decomposition aimed at comparing the stopping region of the considered problem with a region constructed from stopping regions of one-dimensional optimal stopping problems. This decomposition does not use the information on the interaction between the processes but allows to treat any basket containing several types of stochastic processes. Now, if the basket contains several GBMs, Proposition 1 can be refined by decomposing the reward function as a sum of call and put options on assets that are not GBMs and two dimensional exchange options on assets that are GBMs.

In the following, we assume that the basket contains GBMs as well as other Ito diffusions of the $\mathcal{C}$ and $\mathcal{P}$ classes. We write $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ where the index 1 denotes GBM assets, i.e. that X_l is a GBM *if and only if* $l \in I_1 \cup J_1$. We will also note X_{I_1} and X_{J_1} the components of X_I and X_J that are GBMs, i.e. $X_{I_1} = \{X_i, i \in I_1\}$, $X_{J_1} = \{X_j, j \in J_1\}$. To harmonize notation, we use the convention that μ_{i0} and σ_{i0} (resp. μ_{0j} and σ_{0j}) are computed using $\mu_j = \sigma_j = 0$ (resp. $\mu_i = \sigma_i = 0$) in the definitions

(9) and (8) of μ_{ij} and σ_{ij}. Using μ_{i0} and σ_{i0} (resp. μ_{0j} and σ_{0j}) we define naturally $\beta_{i0}^{\pm}$ and γ_{i0} (resp. $\beta_{0j}^{\pm}$ and γ_{0j}) using (6) and (7). Thus if asset i is a GBM, we have

$$C_i(k) = \gamma_i^+ k = \gamma_{i0} k$$

and (in the following we use the equality $\beta_{ij}^- = 1 - \beta_{ji}^+$, which is simple to show)

$$P_j(k) = \gamma_j^- k = \left(\frac{\beta_{j0}^-}{\beta_{j0}^- - 1} \right) k = \left(\frac{1 - \beta_{0j}^+}{-\beta_{0j}^+} \right) k = \frac{1}{\gamma_{0j}} k.$$

In other words $\gamma_i^+ = \gamma_{i0}$ and $\gamma_j^- = 1/\gamma_{0j}$. We establish accurate sufficient conditions for optimal stopping for two particular configurations of the problem that are special cases of the $(1, m)$ and $(n, 1)$ exchange, respectively.

Proposition 3 (Sufficient condition for optimal stopping - $(1, m)$ exchange of a mix for a GBM)**.** *Assume*

1. $n = 1$ *and* $I_1 \neq \emptyset$, $J_1 \neq \emptyset$ *(there is only one income and this income is a GBM);*
2. $X_j \in \mathcal{P}$, $\forall j \in J_2$.

Then $\mathbb{S}_{1,m} \supseteq \mathbb{S}_{1,m}^- \supseteq \mathbb{S}_{1,m}^{--}$ *with*

$$\mathbb{S}_{1,m}^- \triangleq \left\{ x \in \Re^{1+m} : x_1 \geq \sum_{J_1} \gamma_{1j} x_j + \gamma_{10} \sum_{J_2} P_j^{-1}(x_j) \right\}. \tag{40}$$

Recall that we defined the set $\mathbb{S}_{n,m}^-$ by (11) for basket exclusively composed of GBMs and arbitrary values of n and m. In Proposition 3 we have $I_2 = \emptyset$ by assumption while J_2 may or may not be empty. If $J_2 = \emptyset$ in (40) we recover the sufficient condition for optimal stopping obtained by Olsen and Stensland (1992) (see Example 1) which is (11) with $n = 1$. Thus (40) is an extension to hybrid exchanges of the set $\mathbb{S}_{n,m}^-$ defined by (11) for the GBM case.

Proof. Use the decomposition:

$$X_1 - \sum_J X_j = \sum_J (p_j X_1 - X_j) = \sum_{J_1 \cup J_2} (p_j X_1 - X_j)$$

with

$$\sum_J p_j = 1. \tag{41}$$

i) Fix $x \in \Re^{1+m}$. Suppose there exists $p \in \Re_+^m$ such that (41) holds with

$$\begin{aligned} p_j x_1 &\geq \gamma_{1j} x_j \quad \forall j \in J_1, \\ p_j C_1^{-1}(x_1) = p_j x_1/\gamma_{10} &\geq P_j^{-1}(x_j) \quad \forall j \in J_2. \end{aligned} \tag{42}$$

where (42) comes from Example 6, Eq. (36). It is easy to see that x belongs to $\mathbb{S}_{1,m}$.

ii) It remains to determine under which conditions one can construct such a $p \in \Re^m_+$. Choose a particular $t \in J_1$ and define a candidate $\tilde{p}$ for p by:

$$\tilde{p}_j \triangleq \frac{\gamma_{1j} x_j}{x_1}, \quad \forall j \in J_1 \setminus \{t\}$$
$$\tilde{p}_j \triangleq \frac{P_j^{-1}(x_j)}{(x_1/\gamma_{10})}, \quad \forall j \in J_2$$
$$\tilde{p}_t \triangleq 1 - \sum_{J_1 \cup J_2 \setminus \{t\}} \tilde{p}_j.$$

If $\tilde{p}_t x_1 \geq \gamma_{1t} x_t$ i.e. if

$$\left(1 - \sum_{J_1 \setminus \{t\}} \frac{\gamma_{1j} x_j}{x_1} - \sum_{J_2} \gamma_{10} \frac{P_j^{-1}(x_j)}{x_1}\right) x_1 \geq \gamma_{1t} x_t,$$

which is equivalent to

$$x_1 \geq \sum_{J_1} \gamma_{1j} x_j + \gamma_{10} \sum_{J_2} P_j^{-1}(x_j),$$

then the candidate $\tilde{p}$ is in fact such p, and $x \in \mathbb{S}_{1,m}$. We have proved $\mathbb{S}_{1,m} \supseteq \mathbb{S}^-_{1,m}$. It follows from $\gamma_{ij} \leq \gamma_i^+/\gamma_j^- = \gamma_{i0}\gamma_{0j}$ that $\mathbb{S}^-_{1,m} \supseteq \mathbb{S}^{--}_{1,m}$, which completes the proof.

□

The same argument can be used to prove the following result for $(n, 1)$ exchanges.

Proposition 4 (Sufficient condition for optimal stopping - $(n, 1)$ exchange of a GBM for a mix)**.** *Assume*

1. $m = 1$ *and* $I_1 \neq \emptyset$, $J_1 \neq \emptyset$ *(there is only one cost and this cost is a GBM);*
2. $X_i \in \mathcal{C}$, $\forall i \in I_2$.

Then $\mathbb{S}_{n,1} \supseteq \mathbb{S}^-_{n,1} \supseteq \mathbb{S}^{--}_{n,1}$ *with*

$$\mathbb{S}^-_{n,1} \triangleq \left\{ x \in \Re^{n+1} : x_{n+1} \leq \sum_{I_1} \frac{x_i}{\gamma_{i,n+1}} + \frac{1}{\gamma_{0,n+1}} \sum_{I_2} C_i^{-1}(x_i) \right\}.$$

Propositions 3 and 4 will turn out useful in Section 7 where necessary conditions for optimal stopping will be derived.

6 Sufficient conditions when the basket contains GBMs - The general case

6.1 Difficulties

In Section 4 we provided a general sufficient condition for optimal stopping $\mathbb{S}^{--}_{n,m}$ (Subsection 4.1, Proposition 1) but claimed that this condition is probably too strong if the basket only contains GBMs (Subsection 4.2). In Section 5 we gave finer sufficient conditions $\mathbb{S}^-_{1,m} \supseteq \mathbb{S}^{--}_{1,m}$ and $\mathbb{S}^-_{n,1} \supseteq \mathbb{S}^{--}_{n,1}$ for particular configurations of $(1, m)$ and $(n, 1)$

exchanges of baskets containing GBMs. We now show that such analytical refinement of Proposition 1 is difficult to provide for general (n, m) exchanges.

Having in view the determination of a weakest sufficient condition for optimal stopping, it is intuitive to consider the decomposition

$$\sum_I X_i - \sum_J X_j = \sum_{I,J} (c_{ij} X_i - d_{ij} X_j) \tag{43}$$

subject to the conditions

$$\begin{array}{ll} \sum_J c_{ij} = 1 & i \in I, \\ \sum_I d_{ij} = 1 & j \in J, \end{array} \tag{44}$$

and to try to determine a sufficient condition following the reward decomposition technique. We apply the usual two steps approach.

i) Let $x \in \Re^{n+m}$. Suppose there exists c, $d \geq 0$ s.t.

$$\begin{array}{rl} c_{ij} x_i \geq \gamma_{ij} d_{ij} x_j & i \in I_1, j \in J_1, \\ c_{ij} x_i \geq d_{ij} \gamma_{i0} P_j^{-1}(x_j) & i \in I_1, j \in J_2, \\ c_{ij} C_i^{-1}(x_i) \geq d_{ij} \gamma_{0j} x_j & i \in I_2, j \in J_1, \\ c_{ij} C_i^{-1}(x_i) \geq d_{ij} P_j^{-1}(x_j) & i \in I_2, j \in J_2. \end{array} \tag{45}$$

Then $x \in \mathbb{S}_{n,m}$ since (using the decomposition)

$$\begin{aligned} &\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(\sum_I X_i^\tau - \sum_J X_j^\tau \right) \right] \\ &\quad \leq \sum_{k,l=1,2} \sum_{i \in I_k} \sum_{j \in J_l} \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} \left(c_{ij} X_i^\tau - d_{ij} X_j^\tau \right) \right] \\ &\quad = \sum_{k,l=1,2} \sum_{i \in I_k} \sum_{j \in J_l} (c_{ij} x_i - d_{ij} x_j) \\ &\quad = \sum_I x_i - \sum_J x_j = g(x). \end{aligned}$$

ii) It remains to find the positive c and d satisfying conditions (44) and (45) for $x \in \Re^{n+m}$. Note that it is no longer possible to simply fix $n + m - 1$ components of x and deduce a condition on its last component to derive c and d.

We could not complete the step 2 of the reward decomposition technique. Thus we failed to provide an analytic sufficient condition for optimal stopping. Is it really a problem? Not from the practical perspective. Given the current value $x \in \Re^{n+m}$ of assets of its portfolio, the trader only needs an efficient method to determine whether or not he should stop at x. Adopting this point of view—which does not focus on determining an analytic representation of the subset of the stopping region—one can exploit the reward decomposition technique to its maximum effect.

6.2 Linear programming as a remedy

Given some $x \in \Re_+^{n+m}$, linear programming can be used to determine whether or not there exists some positive c, d satisfying the conditions (44) and (45). The problem boils down to verify whether the polyhedron

$$\mathcal{P}_x = \left\{ \begin{array}{c} (c,d) \in \\ \Re_+^{n\times m} \times \Re_+^{n\times m} \end{array} \left| \begin{array}{cc} \sum_J c_{ij} = 1 & i \in I \\ \sum_I d_{ij} = 1 & j \in J \\ c_{ij} x_i \geq \gamma_{ij} d_{ij} x_j & i \in I_1, j \in J_1 \\ c_{ij} x_i \geq \gamma_{i0} d_{ij} P_j^{-1}(x_j) & i \in I_1, j \in J_2 \\ c_{ij} C_i^{-1}(x_i) \geq d_{ij} \gamma_{0j} x_j & i \in I_2, j \in J_1 \\ c_{ij} C_i^{-1}(x_i) \geq d_{ij} P_j^{-1}(x_j) & i \in I_2, j \in J_2 \end{array} \right. \right\} \tag{46}$$

is not empty. Needless to say the problem of determining if a polyhedron is empty or not is solved by linear optimization algorithms. We are thus able to characterize a new subset of the stopping region using linear programming.

Proposition 5 (A LP formulation of the sufficient condition for optimal stopping)**.** *Let* $x \in \Re^{n+m}$ *and* $\mathcal{P}_x$ *be the polyhedron defined by* (46)*. Then* $\mathcal{P}_x \neq \emptyset \Rightarrow x \in \mathbb{S}_{n,m}$*. In other words, defining* $\mathbb{S}_{n,m}^{\mathrm{LP}-} \triangleq \{x \in \Re^{n+m} : \mathcal{P}_x \neq \emptyset\}$ *we have* $\mathbb{S}_{n,m} \supseteq \mathbb{S}_{n,m}^{\mathrm{LP}-}$*.*

Proof. See the part i) of the reward decomposition technique, page 14. □

Recall that we defined $\mathbb{S}_{n,m}^{-}$

- by (11) for exercise of an (n,m) basket of geometric Brownian motions;
- by Proposition 3 (resp. Proposition 4) for exercise of a particular $(1,m)$ (resp. $(n,1)$) heterogeneous basket.

It is clear from Example 6 that if $I_2 = J_2 = \emptyset$ then $\mathbb{S}_{n,m}^{\mathrm{LP}-} = \mathbb{S}_{n,m}^{--}$. We can also make the following straightforward observation.

Corrolary 3 ($\mathbb{S}_{n,m}^{\mathrm{LP}-} = \mathbb{S}_{n,m}^{-}$ in two particular problem configurations)**.**

1. *If* I_1, J_1, $J_2 \neq \emptyset$ *and* $I_2 = \emptyset$ *then* $\mathbb{S}_{1,m}^{\mathrm{LP}-} = \mathbb{S}_{1,m}^{-}$*.*
2. *If* I_1, I_2, $J_1 \neq \emptyset$ *and* $J_2 = \emptyset$ *then* $\mathbb{S}_{n,1}^{\mathrm{LP}-} = \mathbb{S}_{n,1}^{-}$*.*

Proof. We only prove the first point; the proof of the second being similar. If $n = 1$, then the decomposition "(43) subject to (44)" simplifies to $\sum_{I,J}(c_{ij}X_i - d_{ij}X_j) = \sum_J (c_{1j}X_1 - d_{1j}X_j) = \sum_J (c_{1j}X_1 - X_j)$ subject to $\sum_J c_{1j} = 1$ which is the decomposition used to prove Proposition 3. □

6.3 The basket only contains GBMs - Part 2

In particular, if we apply Proposition 5 to a (n,m) basket of GBMs, we have a sufficient condition for optimal stopping:

Example 7 ((n,m) exchange of GBMs)**.** *Assume that the basket only contains geometric Brownian motions. By putting* I_2, $J_2 = \emptyset$ *in* (46) *we have*

$$\mathcal{P}_x = \left\{ (c,d) \in \Re_+^{n\times m} \times \Re_+^{n\times m} \left| \begin{array}{cc} \sum_J c_{ij} = 1 & i \in I \\ \sum_I d_{ij} = 1 & j \in J \\ x_i c_{ij} \geq \gamma_{ij} d_{ij} x_j & i \in I, j \in J \end{array} \right. \right\} \tag{47}$$

and $\mathbb{S}_{n,m} \supseteq \mathbb{S}_{n,m}^{\mathrm{LP}-}$ *where* $\mathbb{S}_{n,m}^{\mathrm{LP}-} \triangleq \{x \in \Re^{n+m} : \mathcal{P}_x \neq \emptyset\}$*.*

In fact, it turns out that in this case $\mathbb{S}^{\text{LP}-}_{n,m} = \mathbb{S}^{-}_{n,m}$. In other words, using the reward decomposition technique on the decomposition (43), one can prove that the convex hull $\mathbb{S}^{-}_{n,m}$ defined by (11) is a subset of the true stopping region $\mathbb{S}_{n,m}$. The reward decomposition technique is thus a unified method to obtain sufficient conditions for optimal stopping.

Proposition 6 ($\mathbb{S}^{\text{LP}-}_{n,m} = \mathbb{S}^{-}_{n,m}$ for baskets of geometric Brownian motions)**.** *Assume that the basket only contains geometric Brownian motions. Then $\mathbb{S}^{-}_{n,m} = \mathbb{S}^{\text{LP}-}_{n,m}$. In other words* (47) *is a LP representation of $\mathbb{S}^{-}_{n,m}$ in the geometric Brownian motion case.*

To prove Proposition 6, we will need an auxiliary result. We made the observation that it is not easy to determine whether a given point $x \in \Re^{n+m}_{+}$ belongs to $\mathbb{S}^{-}_{n,m} = \operatorname{conv}(\bigcup_{i\in I, j\in J} A_{ij})$ using simply the definition of the convex hull. In fact, a non trivial result due to Balas (1998) gives a linear programming representation of the convex hull of the union $\bigcup_i \mathcal{P}_i$ of polyhedras of the form $\mathcal{P}_i \triangleq \{x : M^i x \geq b^i, x \geq 0\}$ with x and b^i vectors and M^i a matrix. In our particular case, this result takes the following very simple form:

Lemma 1 (Balas LP representation of $\mathbb{S}^{-}_{n,m}$)**.**

$$\begin{aligned}
\mathbb{S}^{-}_{n,m} &= \operatorname{conv}\left(\bigcup_{\substack{i\in I\\ j\in J}} A_{ij}\right) \\
&= \left\{ x \in \Re^{n+m} \;\middle|\; \begin{array}{rl} x = \sum_{\substack{i\in I\\ j\in J}} \xi^{ij}, & \xi^{ij} \in \Re^{n+m} \\ \xi^{ij}_i - \gamma_{ij}\xi^{ij}_j \geq 0, & i \in I, j \in J \\ \xi^{ij}_k = 0, & i \in I, j \in J, k \neq i, j \\ \xi^{ij} \geq 0, & i \in I, j \in J \end{array} \right\}.
\end{aligned}$$

Proof. See Balas (1998), Theorem 2.1. □

Lemma 1 is of considerable use in practice. It becomes very simple to check whether a given point $x \in \Re^{n+m}$ belongs to $\mathbb{S}^{-}_{n,m}$: it suffices to check whether the polyhedron

$$\overline{\mathcal{P}}_x \triangleq \left\{ \xi^{ij}, i \in I, j \in J \;\middle|\; \begin{array}{rl} x = \sum_{\substack{i\in I\\ j\in J}} \xi^{ij} & \xi^{ij} \in \Re^{n+m} \\ \xi^{ij}_i - \gamma_{ij}\xi^{ij}_j \geq 0, & i \in I, j \in J \\ \xi^{ij}_k = 0, & i \in I, j \in J, k \neq i, j \\ \xi^{ij} \geq 0, & i \in I, j \in J \end{array} \right\} \tag{48}$$

is empty or not, which can be done easily using any optimization solver. If $\overline{\mathcal{P}}_x \neq \emptyset$ then $x \in \mathbb{S}^{-}_{n,m}$. Let us stress that the particular structure of the sets A_{ij} considerably simplifies the LP representation of the convex hull.[2] In Lemma 1 we in fact have $\xi^{ij} \in A_{ij}$, $i \in I$, $j \in J$ so that one can concisely write

$$\mathbb{S}^{-}_{n,m} = \left\{ x \in \Re^{n+m} \;\middle|\; x = \sum_{\substack{i\in I\\ j\in J}} \xi^{ij}, \quad \xi^{ij} \in A_{ij} \right\} = \operatorname{coni}\left(\bigcup_{\substack{i\in I\\ j\in J}} A_{ij}\right)$$

[2] Only the i-th and j-th components of points of A_{ij} are non zeros. A_{ij} is stable under addition and stable under multiplication by a positive scalar.

where $\mathrm{coni}(S)$ denotes the *conical hull*[3] of a set S (to write the last equality, we have used stability of A_{ij} under addition and multiplication by a positive scalar). Thus we also have

Corrolary 4.

$$\mathbb{S}^{-}_{n,m} = \mathrm{conv}\left(\bigcup_{\substack{i\in I\\ j\in J}} A_{ij}\right) = \mathrm{coni}\left(\bigcup_{\substack{i\in I\\ j\in J}} A_{ij}\right).$$

We are now ready to prove Proposition 6.

Proof.

1. We first prove that $x \in \mathbb{S}^{\mathrm{LP}-}_{n,m} \Rightarrow x \in \mathbb{S}^{-}_{n,m}$. Assume $x \in \mathbb{S}^{\mathrm{LP}-}_{n,m}$. Then $\mathcal{P}_x \neq \emptyset$ with $\mathcal{P}_x$ defined by (47). Thus taking any $(c,d) \in \mathcal{P}_x$, we can define the family of vectors $\{\xi^{ij} \in \Re^{n+m},\, i \in I,\, j \in J\}$ by

$$\begin{aligned} \xi^{ij}_i &= c_{ij}x_i \\ \xi^{ij}_j &= d_{ij}x_j \\ \xi^{ij}_k &= 0 \qquad \forall k \neq i,j. \end{aligned}$$

   It is simple algebra to prove that the family of vectors $\{\xi^{ij} \in \Re^{n+m},\, i \in I,\, j \in J\}$ belongs to $\overline{\mathcal{P}_x}$ which is therefore not empty. Thus $x \in \mathbb{S}^{-}_{n,m}$.

2. We then prove $x \in \mathbb{S}^{-}_{n,m} \Rightarrow x \in \mathbb{S}^{\mathrm{LP}-}_{n,m}$. Assume $x \in \mathbb{S}^{-}_{n,m}$. Then $\overline{\mathcal{P}_x} \neq \emptyset$ with $\overline{\mathcal{P}_x}$ defined by (48). Thus taking any $\xi \in \overline{\mathcal{P}_x}$, we can define the two matrices $\{c_{ij}\}$, $\{d_{ij}\} \in \Re^{n\times m}$ by

$$\begin{aligned} c_{ij} &= \xi^{ij}_i/x_i \quad &&\text{if} \quad x_i \neq 0, \\ c_{ij} &= 0 \quad &&\text{if} \quad x_i = 0, \\ d_{ij} &= \xi^{ij}_j/x_j \quad &&\text{if} \quad x_j \neq 0, \\ d_{ij} &= 0 \quad &&\text{if} \quad x_j = 0. \end{aligned}$$

   It is simple algebra to prove that the defined family c and d belong to $\mathcal{P}_x$ which is therefore not empty. Thus $x \in \mathbb{S}^{\mathrm{LP}-}_{n,m}$.

□

To sum up, we have proved that the reward decomposition technique is a unified method to obtain sufficient conditions for optimal stopping:

1. For very general exchange problems, it allows to numerically characterize a subset of the stopping region through linear programming (e.g. Propositions 5 and 6);

2. When this subset of the stopping region is analytically determinable, it provides its analytic expression (e.g. Example 1 and Propositions 1, 3, 4).

The next section is devoted to the second type of characterization of the stopping region: we now want to determine a necessary condition for optimal stopping i.e. a set containing the stopping region $\mathbb{S}_{n,m}$.

[3] The conical hull $\mathrm{coni}(S)$ of a set S is defined by

$$\mathrm{coni}(S) \triangleq \left\{ \sum_{i=1}^{n} c_i x_i \,\middle|\, x_i \in S, c_i \in \Re \text{ and } c_i > 0 \quad \forall i = 1,\dots,n, \quad \forall n \in \mathbb{N} \right\}.$$

7 Necessary conditions for optimal stopping

This section provides necessary conditions for the optimal stopping Problem 1. The result is inspired by works of Hu and Øksendal (1998) and Nishide and Rogers (2011) and is obtained from the following reasoning. Suppose that x belongs to the stopping region of $\sup_{\tau\in\mathcal{S}} \mathbb{E}^x\left[\sum_I X_i - \sum_J X_j\right]$. Introduce two arbitrary chosen geometric Brownian motions X_u and X_v. If it is optimal to simultaneously proceed to the two exchanges

1. exchange of X_v for $\sum_J X_j$;
2. exchange of $\sum_I X_i$ for X_u;

then, since it is optimal to exchange $\sum_J X_j$ for $\sum_I X_i$, it is necessarily optimal to exchange immediately X_u for X_v.

Proposition 7 (Necessary condition for optimal stopping)**.**

1. *Assume that the stochastic vector $X(t,\omega) \in \Re^{n+m}$ satisfies the usual conditions:*
   (a) either $I_1,\ J_1 \neq \emptyset$, or $I_1,\ J_1 = \emptyset$;
   (b) $\forall i \in I_2,\ X_i \in \mathcal{C}$;
   (c) $\forall j \in J_2,\ X_j \in \mathcal{P}$.
2. *In addition, choose two arbitrary geometric Brownian motions X_u and X_v, possibly correlated.*

Then $\mathbb{S}_{n,m} \subseteq \mathbb{S}^+_{n,m}(X_u, X_v)$ with

$$\mathbb{S}^+_{n,m}(X_u,X_v) \triangleq \Bigg\{ x \in \Re^{n+m}_+ : \sum_{I_1} \gamma_{ui} x_i + \gamma_{u0} \sum_{I_2} P_i^{-1}(x_i) \geq \gamma_{uv} \left(\sum_{J_1} \frac{x_j}{\gamma_{jv}} + \frac{1}{\gamma_{0v}} \sum_{J_2} C_i^{-1}(x_j) \right) \Bigg\}. \tag{49}$$

Note that we defined the set $\mathbb{S}^+_{n,m}$ by (14) for basket exclusively composed of GBMs and that we recover (14) in Proposition 7 if $I_2 = J_2 = \emptyset$. Thus (49) is an extension to hybrid exchanges of the set $\mathbb{S}^+_{n,m}$ defined by (14) for the GBM case.

Proof. Let us recall that the stopping region of a discounted time homogeneous optimal problem $\sup_{\tau\in\mathcal{S}} \mathbb{E}^x[e^{-r\tau} g(X_\tau)]$ is noted $\mathbb{S}[g]$. Since X_u and X_v are both GBMs, we have

$$\mathbb{S}\left[X_u - X_v\right] = \left\{ x_u, x_v \in \Re_+ : x_v \geq \gamma_{uv} x_v \right\}. \tag{50}$$

By Propositions 3 and 4, we also have

$$\mathbb{S}\left[\sum_J X_j - X_v\right] \supseteq \mathbb{S}^-_{m,1} \triangleq \left\{ x_v \leq \sum_{J_1} \frac{x_j}{\gamma_{jv}} + \frac{1}{\gamma_{0v}} \sum_{J_2} C_j^{-1}(x_j) \right\} \tag{51}$$

$$\mathbb{S}\left[X_u - \sum_I X_i\right] \supseteq \mathbb{S}^-_{1,n} \triangleq \left\{ x_u \geq \sum_{I_1} \gamma_{ui} x_i + \gamma_{u0} \sum_{I_2} P_i^{-1}(x_i) \right\}. \tag{52}$$

i) Suppose $x \in \mathbb{S}_{n,m}$.

ii) Choose any geometric Brownian motion X_v. Define

$$\mathbb{S}_v \triangleq \left\{ (x, x_v) \middle| x \in \mathbb{S}_{n,m} \, , \, (x, x_v) \in \mathbb{S}^-_{m,1} \right\}.$$

Using the reward decomposition technique, it is easy to prove that if $(x, x_v) \in \mathbb{S}_v$, then $(x, x_v) \in \mathbb{S}[\sum_I X_i - X_v]$: If $(x, x_v) \in \mathbb{S}_v$ then

$$\begin{aligned}
&\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,x_v} \left[e^{-r\tau} \left(\sum_I X_i^\tau - X_v^\tau \right) \right] \\
&\quad = \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,x_v} \left[e^{-r\tau} \left(\sum_I X_i^\tau - \sum_J X_j^\tau \right) + e^{-r\tau} \left(\sum_J X_j^\tau - X_v^\tau \right) \right] \\
&\quad \leq \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x} \left[e^{-r\tau} \left(\sum_I X_i^\tau - \sum_J X_j^\tau \right) \right] + \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,x_v} \left[e^{-r\tau} \left(\sum_J X_j^\tau - X_v^\tau \right) \right] \\
&\quad \leq \left(\sum_I x_i - \sum_J x_j \right) + \left(\sum_J x_j - x_v \right) = \sum_I x_i - x_v.
\end{aligned}$$

iii) Now take any geometric Brownian motion X_u. Define

$$\mathbb{S}_{u,v} \triangleq \left\{ (x_u, x, x_v) \middle| (x, x_v) \in \mathbb{S}_v \, , \, (x_u, x) \in \mathbb{S}^-_{1,n} \right\}.$$

It is also easy to prove that if $(x_u, x, x_v) \in \mathbb{S}_{u,v}$, then $(x_u, x, x_v) \in \mathbb{S}[X_u - X_v]$: Choose $(x_u, x, x_v) \in \mathbb{S}_{u,v}$. We see that

$$\begin{aligned}
&\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x_u,x,x_v} \left[e^{-r\tau} (X_u^\tau - X_v^\tau) \right] \\
&\quad \leq \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x_u,x} \left[e^{-r\tau} \left(X_u^\tau - \sum_I X_i^\tau \right) \right] + \sup_{\tau \in \mathcal{S}} \mathbb{E}^{x,x_v} \left[e^{-r\tau} \left(\sum_I X_i^\tau - X_v^\tau \right) \right] \\
&\quad \leq \left(x_u - \sum_I x_i \right) + \left(\sum_I x_i - x_v \right) = x_u - x_v.
\end{aligned}$$

iv) We have thus proved that, for any GBMs X_u and X_v, if $x \in \mathbb{S}_{n,m}$, $(x, x_v) \in \mathbb{S}_v$ and $(x_u, x, x_v) \in \mathbb{S}_{u,v}$, then $(x_u, x_v) \in \mathbb{S}[X_u - X_v]$ i.e. (50) holds. We can thus construct a necessary condition for optimal stopping, using (51) and (52): for any $x \in \mathbb{S}_{n,m}$, define

$$\begin{aligned}
\tilde{x}_u &\triangleq \sum_{I_1} \gamma_{ui} x_i + \gamma_{u0} \sum_{I_2} P_i^{-1}(x_i), \\
\tilde{x}_v &\triangleq \sum_{J_1} \frac{x_j}{\gamma_{jv}} + \frac{1}{\gamma_{0v}} \sum_{J_2} C_j^{-1}(x_j).
\end{aligned}$$

We have by construction $x \in \mathbb{S}_{n,m}$, $(x, \tilde{x}_v) \in \mathbb{S}_v$ (by Eq. 51) and $(\tilde{x}_u, x, \tilde{x}_v) \in \mathbb{S}_{uv}$ (by Eq. 52). We should therefore have $\tilde{x}_u \geq \gamma_{uv} \tilde{x}_v$ (by Eq. 50) which is

$$\left(\sum_{I_1} \gamma_{ui} x_i + \gamma_{u0} \sum_{I_2} P_i^{-1}(x_i) \right) \geq \gamma_{uv} \left(\sum_{J_1} \frac{x_j}{\gamma_{jv}} + \frac{1}{\gamma_{0v}} \sum_{J_2} C_j^{-1}(x_j) \right),$$

and the proof is complete.

$\square$

Table 1 summarizes our results. The next section provides illustrating numerical examples.

	Exchange configuration	References	Sufficient	Necessary
GBMs	$n = m = 1$	MDS(89)	Solved	Solved
	n, m given	OS(92) HØ(98) NR(11) this paper	$[\mathbb{S}^{\mathrm{LP}-}_{n,m} = \mathbb{S}^{-}_{n,m}]$ $^{(1)}$	$\mathbb{S}^{+}_{n,m}$
		GS(11)	$\mathbb{S}^{-}_{n,m} \subseteq \mathbb{S}^{\diamond}_{n,m} \subseteq \mathbb{S}^{+}_{n,m}$ $^{(2)}$	
Mix	$n = 1$ $X_1 \sim$ GBM	this paper	$\mathbb{S}^{\mathrm{LP}-}_{1,m} = \mathbb{S}^{-}_{1,m}$	$\mathbb{S}^{+}_{1,m}$
	$m = 1$ $X_{n+1} \sim$ GBM		$\mathbb{S}^{\mathrm{LP}-}_{n,1} = \mathbb{S}^{-}_{n,1}$	$\mathbb{S}^{+}_{n,1}$
	n, m arbitrary no GBMs		$\mathbb{S}^{\mathrm{LP}-}_{n,m} = \mathbb{S}^{--}_{n,m}$	$\mathbb{S}^{+}_{n,m}$
	n, m arbitrary		$[\mathbb{S}^{\mathrm{LP}-}_{n,m}]$	

Table 1: A synthesis of the results. References: MDS(89) = McDonald and Siegel (1986), OS(92) = Olsen and Stensland (1992), HØ(98) = Hu and Øksendal (1998), GS(11) = Gahungu and Smeers (2011). Recall that necessary conditions usually depend on arbitrary chosen GBMs (2 in general, 1 for $(1, m)$ and $(n, 1)$ exchanges). The sets under brackets can only be characterized numerically. $^{(1)}$Yields an analytic expression for $n = 1$ or $m = 1$, needs numerical characterization otherwise. $^{(2)}$On selected examples (see Gahungu and Smeers, 2011).

8 Numerical examples

8.1 Exchange of a GBM and a GMRP for a Schwartz process

Consider the $(1, 2)$ exchange under the setting $X_1 \sim \mathrm{schw}(0.03, 0.4)$, $X_2 \sim \mathrm{gbm}(0.02, 0.1)$, $X_3 \sim \mathrm{gmpr}(0.05, 0.2, 0.2)$, $r = 0.1$. Note that the only sufficient condition for optimal stopping one can give here—$\mathbb{S}^{--}_{1,2}$ given by Proposition 1—do not depend on assets correlations, which is quite symptomatic of its lack of precision. Therefore, we work without any assumptions on assets correlation. In further examples if no values are provided for some correlations, this indicates that they do not enter into the rules to be used.

Figure 1 represents in continuous line the boundary $\mathrm{d}\mathbb{S}^{--}_{1,2}$ of the subset $\mathbb{S}^{--}_{1,2} \subset \mathbb{S}_{1,2}$ in the plan $x_2 = 4$ ($\mathbb{S}^{--}_{1,2}$ is above $\mathrm{d}\mathbb{S}^{--}_{1,2}$). $\mathbb{S}^{--}_{1,2}$ were determined using Proposition 1. The dotted line gives the natural net present value (positive reward) condition $x_1 \geq x_2 + x_3$.

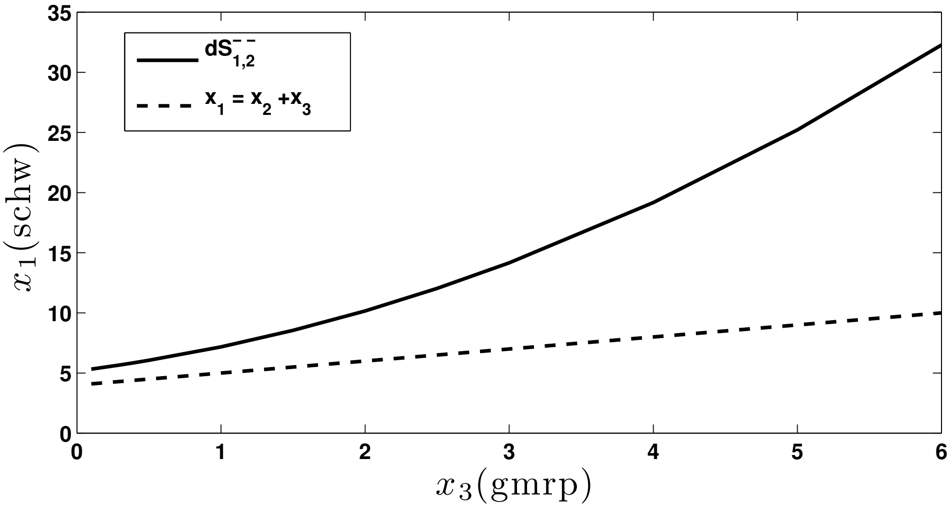

Figure 1: Exchange of a geometric Brownian motion plus a geometric mean reverting process for a Schwartz process. The figure represents the sufficient condition $\mathbb{S}^{--}_{1,2}$ in the plan $x_2 = 4$ as a function of the second cost x_3.

8.2 Exchange of a GBM and a GMRP for a GBM

Consider the $(1,2)$ exchange under the setting $X_1 \sim \text{gbm}(0.05, 0.2)$, $X_2 \sim \text{gbm}(0.02, 0.1)$, $X_3 \sim \text{gmrp}(0.05, 0.2, 0.2)$, $\rho_{12} = 0.2$, $r = 0.1$. Figure 2 represents the boundaries $d\mathbb{S}^{--}_{1,2}$ and $d\mathbb{S}^{-}_{1,2}$ of the subsets $\mathbb{S}^{--}_{1,2} \subseteq \mathbb{S}_{1,2}$ and $\mathbb{S}^{-}_{1,2} \subseteq \mathbb{S}_{1,2}$, respectively ($\mathbb{S}^{--}_{1,2}$ and $\mathbb{S}^{-}_{1,2}$ are above $d\mathbb{S}^{--}_{1,2}$ and $d\mathbb{S}^{-}_{1,2}$). $\mathbb{S}^{--}_{1,2}$ were determined using Proposition 1, $\mathbb{S}^{-}_{1,2}$ using Proposition 3. We also plotted the positive reward condition $x_1 > x_2 + x_3$ and the strictest necessary condition $d\mathbb{S}^{+}_{1,2}$ one could find using 1000000 randomly generated geometric Brownian motion X_u in Proposition 7.[4] We see that $\mathbb{S}^{--}_{1,2} \subset \mathbb{S}^{-}_{1,2}$ i.e. $\mathbb{S}^{-}_{1,2}$ is the most precise sufficient condition for optimal stopping. The stopping region $\mathbb{S}_{1,2}$ of the problem is such that $\mathbb{S}^{--}_{1,2} \subset \mathbb{S}^{-}_{1,2} \subset \mathbb{S}_{1,2} \subset \mathbb{S}^{+}_{1,2} \subset \{x \in \Re^3_+ : x_1 > x_2 + x_3\}$. Note finally that there is a consequent gap between the sufficient condition $\mathbb{S}^{-}_{1,2}$ and the necessary condition $\mathbb{S}^{+}_{1,2}$.

[4] See Gahungu and Smeers (2011), Appendix D for a description of the methodology used to generate these different geometric Brownian motions.

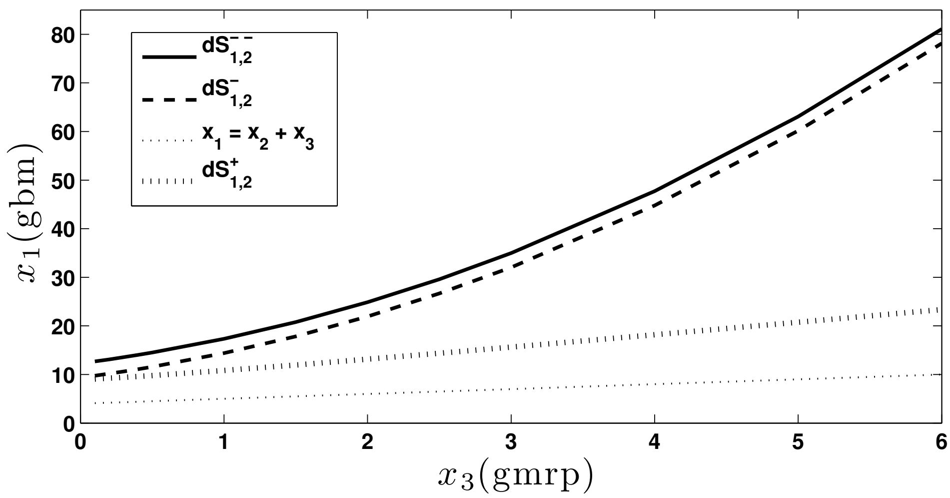

Figure 2: Exchange of a geometric Brownian motion plus a geometric mean reverting process for a geometric Brownian motion. The figure represents the sufficient conditions $\mathbb{S}^{--}_{1,2}$ and $\mathbb{S}^{-}_{1,2}$ and the necessary condition $\mathbb{S}^{+}_{1,2}$ in the plan $x_2 = 4$ as a function of the second cost x_3. Note that $\mathbb{S}^{--}_{1,2} \subset \mathbb{S}^{-}_{1,2} \subset \mathbb{S}^{+}_{1,2}$.

8.3 Exchange of two GBMs for two others

Consider the $(2,2)$ exchange of 4 geometric Brownian motions under the following setting. $X_1 \sim \text{gbm}(0.1, 0.4)$, $X_2 \sim \text{gbm}(0.06, 0.1)$, $X_3 \sim \text{gbm}(0.035, 0.15)$ and $X_4 \sim \text{gbm}(0.12, 0.3)$. The correlations between the driving Brownian motions are $\rho_{12} = 0.25$, $\rho_{13} = 0.35$, $\rho_{14} = -0.5$, $\rho_{23} = -0.25$, $\rho_{24} = 0.2$ and $\rho_{34} = -0.55$. The discount rate is $r = 0.3$. Figure 3 gives the boundary of the following sets.

- The subset of the stopping region $\mathbb{S}^{--}_{2,2} \subseteq \mathbb{S}_{2,2}$ given by Proposition 1.
- The subset of the stopping region

$$\mathbb{S}^{-}_{2,2} = \text{conv}\left(\bigcup_{\substack{i=1,2\\j=3,4}} A_{ij}\right) = \text{coni}\left(\bigcup_{\substack{i=1,2\\j=3,4}} A_{ij}\right) = \mathbb{S}^{\text{LP}-}_{2,2} \tag{53}$$

  given by Proposition 5.
- The stopping region $\mathbb{S}^{\diamond}_{2,2}$ proposed by Gahungu and Smeers (2011, Proposition 3).
- The smaller superset of the stopping region $\mathbb{S}^{+}_{2,2}$ one found using a large number (1000000) of randomly generated GBMs X_u and X_v in (14).

Figure 3 shows that $\mathbb{S}^{--}_{2,2} \subset \mathbb{S}^{-}_{2,2} = \mathbb{S}^{\text{LP}-}_{2,2} \subset \mathbb{S}^{\diamond}_{2,2} \subset \mathbb{S}^{+}_{2,2}$. We see that the gap between $\mathbb{S}^{-}_{2,2}$ and $\mathbb{S}^{+}_{2,2}$ is big, and that this example do not reject the hypothesis that the stopping region $\mathbb{S}^{\diamond}_{2,2}$ introduced by Gahungu and Smeers (2011) might be the optimal stopping region.

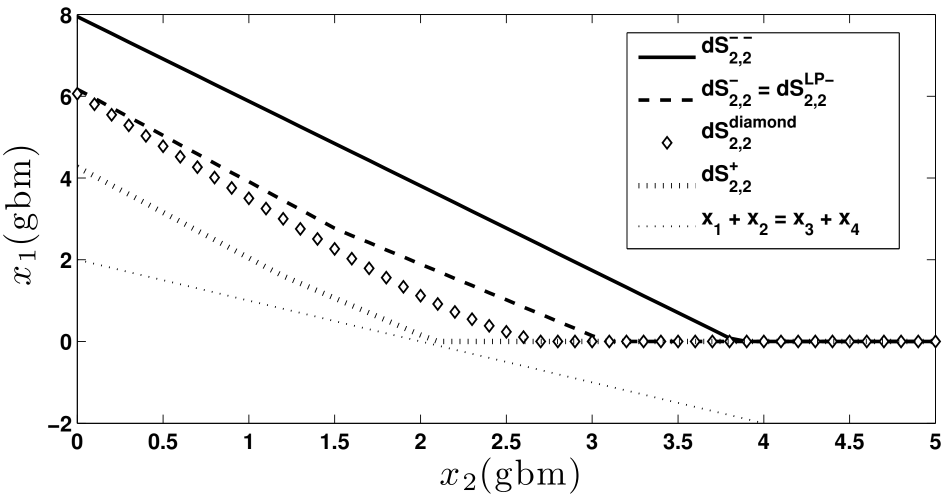

Figure 3: Exchange of two geometric Brownian motions for two others. The figure represents the sufficient conditions $\mathbb{S}^{--}_{2,2}$, $\mathbb{S}^{-}_{2,2} = \mathbb{S}^{\mathrm{LP}-}_{2,2}$, the necessary condition $\mathbb{S}^{+}_{2,2}$ and the investment rule $\mathbb{S}^{\diamond}_{2,2} = \mathbb{S}^{\mathrm{diamond}}_{2,2}$ proposed by Gahungu and Smeers (2011) in the plan $x_3 = x_4 = 1$ as a function of the second price x_2.

8.4 A (2,3) exchange

Consider the $(2,3)$ exchange of 2 geometric Brownian motions for 2 geometric Brownian motions plus a Schwartz process under the following setting. $X_1 \sim \mathrm{gbm}(0.1, 0.4)$, $X_2 \sim \mathrm{gbm}(0.06, 0.1)$, $X_3 \sim \mathrm{gbm}(0.035, 0.15)$, $X_4 \sim \mathrm{gbm}(0.12, 0.3)$ and $X_5 \sim \mathrm{schw}(0.05, 0.2, 0.2)$. The correlations between the driving Brownian motions are $\rho_{12} = 0.25$, $\rho_{13} = 0.35$, $\rho_{14} = -0.5$, $\rho_{23} = -0.25$, $\rho_{24} = 0.2$ and $\rho_{34} = -0.55$. The discount rate is $r = 0.3$. Figure 4 gives the boundary of the following sets.

- The subset of the stopping region $\mathbb{S}^{--}_{2,3} \subseteq \mathbb{S}_{2,3}$ given by Proposition 1.
- The subset of the stopping region $\mathbb{S}^{\mathrm{LP}-}_{2,3}$ given by Proposition 5.
- The smaller superset of the stopping region $\mathbb{S}^{+}_{2,3}$ one found using a large number (1000000) of randomly generated GBMs X_u and X_v in Proposition 7.

Figure 4 shows that $\mathbb{S}^{--}_{2,3} \subset \mathbb{S}^{\mathrm{LP}-}_{2,3} \subset \mathbb{S}^{+}_{2,3}$.

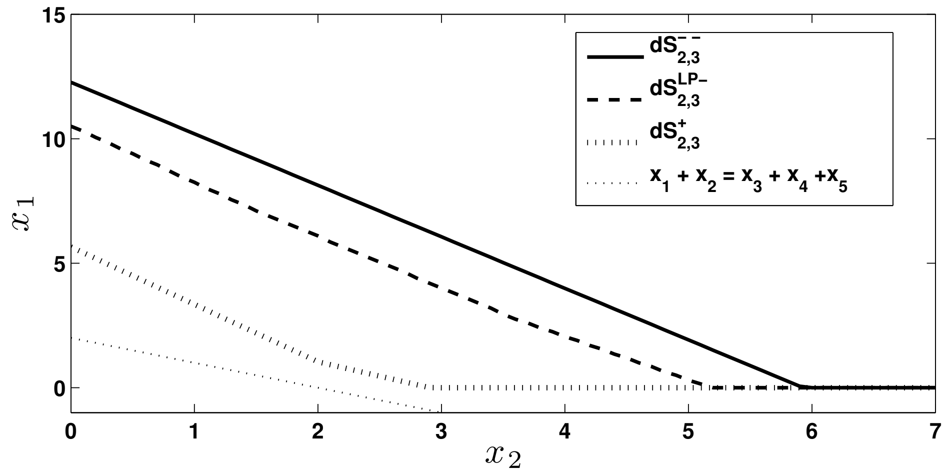

Figure 4: Exchange of two geometric Brownian motions for two others plus a Schwartz process. The figure represents the sufficient conditions $\mathbb{S}^{--}_{2,3}$, $\mathbb{S}^{\mathrm{LP}-}_{2,3}$ and the necessary condition $\mathbb{S}^{+}_{2,3}$ in the plan $x_3 = x_4 = x_5 = 1$ as a function of the second price x_2.

9 Conclusion

In this paper, we have considered the problem of optimal exercise of a perpetual basket option containing several types of Ito-diffusions. When the basket exclusively contains geometric Brownian motions, the problem is known to be already difficult to solve: for now the theory only allows to identify subset and superset of the stopping region (i.e. sufficient conditions and necessary conditions for optimal stopping).

We have provided similar sufficient conditions and necessary conditions for optimal exercise of a hybrid basket option. More precisely, we use systematically the reward decomposition argument introduced by Olsen and Stensland (1992) to derive sufficient conditions for optimal stopping. This technique is a unified method to obtain sufficient conditions for optimal stopping: (i) for very general exchange problems, it allows to characterize a subset of the stopping region numerically through linear programming; (ii) when this subset of the stopping region is analytically determinable, it provides its analytic expression.

In particular, if the basket exclusively contains geometric Brownian motions, the reward decomposition technique is easier to use than the known analytic sufficient condition which is a convex hull of an union of polyhedra (see Nishide and Rogers, 2011).

If sufficient conditions are determinable completely as functions of drifts and volatility rates, they do not depend on all the correlations between assets of the basket. This is a clear indication that these conditions are too strong. Necessary conditions are conditional on two auxiliary processes that have to be chosen arbitrarily. Moreover, they suffer from the same partial correlation invariance, thus are clearly too weak. It is visible on proposed examples that the gap between sufficient and necessary conditions for optimal stopping may be large. This indicates that this gap might also be large in practical cases.

References

Balas, E., 1998. Disjunctive programming: properties of the convex hull of feasible points. Discrete Applied Mathematics 89, 3 – 44.

Boyarchenko, S., Levendorskii, S., 2007. Irreversible decisions under uncertainty. Optimal stopping made easy. Springer.

Dixit, A. K., Pindyck, R. S., January 1994. Investment under Uncertainty. Princeton University Press.

Gahungu, J. M., Smeers, Y., July 2011. Optimal time to invest when the price processes are geometric Brownian motions: a tentative based on smooth fit. CORE Discussion Paper 2011/34, Université catholique de Louvain, Belgium.

Hu, Y., Øksendal, B., 1998. Optimal time to invest when the price processes are geometric Brownian motions. Finance and Stochastics 2, 295 –310.

McDonald, R. L., Siegel, D. R., 1986. The value of waiting to invest. The Quarterly Journal of Economics 101 (4), 707–728.

Metcalf, G. E., Hassett, K. A., 1995. Investment under alternative return assumptions: comparing random walks and mean reversion. Journal of Economic Dynamics and Control 19, 1471 – 1488.

Nishide, K., Rogers, L., March 2011. Optimal time to exchange two baskets. Journal of Applied Probabilities 48 (1), 21–30.

Olsen, T. E., Stensland, G., 1992. On optimal timing of investment when cost components are additive and follow geometric diffusions. Journal of Economic Dynamics and Control 16, 39 – 51.

Recent titles

CORE Discussion Papers

2010/79. Michel LE BRETON, Juan D. MORENO-TERNERO, Alexei SAVVATEEV and Shlomo WEBER. Stability and fairness in models with a multiple membership.
2010/80. Juan D. MORENO-TERNERO. Voting over piece-wise linear tax methods.
2010/81. Jean HINDRIKS, Marijn VERSCHELDE, Glenn RAYP and Koen SCHOORS. School tracking, social segregation and educational opportunity: evidence from Belgium.
2010/82. Jean HINDRIKS, Marijn VERSCHELDE, Glenn RAYP and Koen SCHOORS. School autonomy and educational performance: within-country evidence.
2010/83. Dunia LOPEZ-PINTADO. Influence networks.
2010/84. Per AGRELL and Axel GAUTIER. A theory of soft capture.
2010/85. Per AGRELL and Roman KASPERZEC. Dynamic joint investments in supply chains under information asymmetry.
2010/86. Thierry BRECHET and Pierre M. PICARD. The economics of airport noise: how to manage markets for noise licenses.
2010/87. Eve RAMAEKERS. Fair allocation of indivisible goods among two agents.
2011/1. Yu. NESTEROV. Random gradient-free minimization of convex functions.
2011/2. Olivier DEVOLDER, François GLINEUR and Yu. NESTEROV. First-order methods of smooth convex optimization with inexact oracle.
2011/3. Luc BAUWENS, Gary KOOP, Dimitris KOROBILIS and Jeroen V.K. ROMBOUTS. A comparison of forecasting procedures for macroeconomic series: the contribution of structural break models.
2011/4. Taoufik BOUEZMARNI and Sébastien VAN BELLEGEM. Nonparametric Beta kernel estimator for long memory time series.
2011/5. Filippo L. CALCIANO. The complementarity foundations of industrial organization.
2011/6. Vincent BODART, Bertrand CANDELON and Jean-François CARPANTIER. Real exchanges rates in commodity producing countries: a reappraisal.
2011/7. Georg KIRCHSTEIGER, Marco MANTOVANI, Ana MAULEON and Vincent VANNETELBOSCH. Myopic or farsighted? An experiment on network formation.
2011/8. Florian MAYNERIS and Sandra PONCET. Export performance of Chinese domestic firms: the role of foreign export spillovers.
2011/9. Hiroshi UNO. Nested potentials and robust equilibria.
2011/10. Evgeny ZHELOBODKO, Sergey KOKOVIN, Mathieu PARENTI and Jacques-François THISSE. Monopolistic competition in general equilibrium: beyond the CES.
2011/11. Luc BAUWENS, Christian HAFNER and Diane PIERRET. Multivariate volatility modeling of electricity futures.
2011/12. Jacques-François THISSE. Geographical economics: a historical perspective.
2011/13. Luc BAUWENS, Arnaud DUFAYS and Jeroen V.K. ROMBOUTS. Marginal likelihood for Markov-switching and change-point GARCH models.
2011/14. Gilles GRANDJEAN. Risk-sharing networks and farsighted stability.
2011/15. Pedro CANTOS-SANCHEZ, Rafael MONER-COLONQUES, José J. SEMPERE-MONERRIS and Oscar ALVAREZ-SANJAIME. Vertical integration and exclusivities in maritime freight transport.
2011/16. Géraldine STRACK, Bernard FORTZ, Fouad RIANE and Mathieu VAN VYVE. Comparison of heuristic procedures for an integrated model for production and distribution planning in an environment of shared resources.
2011/17. Juan A. MAÑEZ, Rafael MONER-COLONQUES, José J. SEMPERE-MONERRIS and Amparo URBANO Price differentials among brands in retail distribution: product quality and service quality.
2011/18. Pierre M. PICARD and Bruno VAN POTTELSBERGHE DE LA POTTERIE. Patent office governance and patent system quality.
2011/19. Emmanuelle AURIOL and Pierre M. PICARD. A theory of BOT concession contracts.
2011/20. Fred SCHROYEN. Attitudes towards income risk in the presence of quantity constraints.

Recent titles

CORE Discussion Papers - continued

2011/21. Dimitris KOROBILIS. Hierarchical shrinkage priors for dynamic regressions with many predictors.
2011/22. Dimitris KOROBILIS. VAR forecasting using Bayesian variable selection.
2011/23. Marc FLEURBAEY and Stéphane ZUBER. Inequality aversion and separability in social risk evaluation.
2011/24. Helmuth CREMER and Pierre PESTIEAU. Social long term care insurance and redistribution.
2011/25. Natali HRITONENKO and Yuri YATSENKO. Sustainable growth and modernization under environmental hazard and adaptation.
2011/26. Marc FLEURBAEY and Erik SCHOKKAERT. Equity in health and health care.
2011/27. David DE LA CROIX and Axel GOSSERIES. The natalist bias of pollution control.
2011/28. Olivier DURAND-LASSERVE, Axel PIERRU and Yves SMEERS. Effects of the uncertainty about global economic recovery on energy transition and CO_2 price.
2011/29. Ana MAULEON, Elena MOLIS, Vincent J. VANNETELBOSCH and Wouter VERGOTE. Absolutely stable roommate problems.
2011/30. Nicolas GILLIS and François GLINEUR. Accelerated multiplicative updates and hierarchical als algorithms for nonnegative matrix factorization.
2011/31. Nguyen Thang DAO and Julio DAVILA. Implementing steady state efficiency in overlapping generations economies with environmental externalities.
2011/32. Paul BELLEFLAMME, Thomas LAMBERT and Armin SCHWIENBACHER. Crowdfunding: tapping the right crowd.
2011/33. Pierre PESTIEAU and Gregory PONTHIERE. Optimal fertility along the lifecycle.
2011/34. Joachim GAHUNGU and Yves SMEERS. Optimal time to invest when the price processes are geometric Brownian motions. A tentative based on smooth fit.
2011/35. Joachim GAHUNGU and Yves SMEERS. Sufficient and necessary conditions for perpetual multi-assets exchange options.

Books

P. VAN HENTENRYCKE and L. WOLSEY (eds.) (2007), *Integration of AI and OR techniques in constraint programming for combinatorial optimization problems*. Berlin, Springer.

P-P. COMBES, Th. MAYER and J-F. THISSE (eds.) (2008), *Economic geography: the integration of regions and nations*. Princeton, Princeton University Press.

J. HINDRIKS (ed.) (2008), *Au-delà de Copernic: de la confusion au consensus* ? Brussels, Academic and Scientific Publishers.

J-M. HURIOT and J-F. THISSE (eds) (2009), *Economics of cities.* Cambridge, Cambridge University Press.

P. BELLEFLAMME and M. PEITZ (eds) (2010), *Industrial organization: markets and strategies*. Cambridge University Press.

M. JUNGER, Th. LIEBLING, D. NADDEF, G. NEMHAUSER, W. PULLEYBLANK, G. REINELT, G. RINALDI and L. WOLSEY (eds) (2010), *50 years of integer programming, 1958-2008: from the early years to the state-of-the-art*. Berlin Springer.

G. DURANTON, Ph. MARTIN, Th. MAYER and F. MAYNERIS (eds) (2010), *The economics of clusters – Lessons from the French experience*. Oxford University Press.

J. HINDRIKS and I. VAN DE CLOOT (eds) (2011), *Notre pension en heritage*. Itinera Institute.

CORE Lecture Series

D. BIENSTOCK (2001), Potential function methods for approximately solving linear programming problems: theory and practice.

R. AMIR (2002), Supermodularity and complementarity in economics.

R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.